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ON THE RATLIFF-RUSH CLOSURE OF IDEALS

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This talk will describe some interesting results about Ratliff-Rush closure of ideals. The important idea of the integral closure of an ideal in a commutative Noetherian ring R (with non-zero identity) was introduced by Northcott and Rees [10]. This concept has been extended to ideals in an arbitrary commutative ring by L.J. Ratliff and D. Rush. Namely, the important notion of Ratliff-Rush closure of an ideal in a commutative ring R was introduced and studied in [11] and [12] as a refinement of the integral closure of an ideal, and this new idea has been proved useful in several questions, for example see [4], [5], [7], [14]. It is appropriate for us to provide a brief review. Let R be a commutative ring (with nonzero identity) and let I be an ideal of R. In [12] the interesting ideal,

$$\widetilde{I} := \bigcup_{n\in\mathbb{N}}(I^{n+1} :_R I^n) = \{x \in R \mid xI^n \subseteq I^{n+1} \text{ for some } n \geq 1\}$$

of R, associated with I, has introduced by Ratliff and Rush. If $grade\, I > 0$, then this new ideal has some nice properties, for instance,

$$\text{for all sufficiently large } \widetilde{I^n} = I^n. \tag{0.1}$$

The Ratliff-Rush closure of an ideal I has been studied in [3], [4], [5], [6], [7], [13], [14] and has led to some interesting results.
For an arbitrary non-zero finitely generated module M over a commutative Noetherian ring R, we define the Ratliff-Rush closure $\widetilde{I}^{(M)}$ of an ideal I of R with respect to M and we show that if $grade\,(I, M) > 0$, then

$$\widetilde{I}^{(M)} \supseteq \widetilde{I^2}^{(M)} \supseteq \cdots \supseteq \widetilde{I^n}^{(M)} = I^nM :_R M \text{ for all large } n.$$

Also, we will obtain a finiteness result about the asymptotic prime divisors, namely it is shown that for any ideal I of R the sequence $\{\mathrm{Ass}_R R/\widetilde{I^n}^{(M)}\}_{n\geq 1}$ of associated primes is increasing and eventually stabilizes. Finally, we show that whenever I and J are ideals of R such that $I \subseteq J \subseteq I_a^{(M)}$ and grade $(I, M) > 0$,

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then $\widetilde{I}^{(M)} = \widetilde{J}^{(M)}$. Furthermore, if R is local and I is a Ratliff-Rush reduction of J with respect to M, then I contains a minimal Ratliff-Rush reduction K of J and every minimal basis of K can be extended to a minimal basis of I and the operation $I \to \widetilde{I}^{(M)}$ is a c^*-operation on the set of ideals I of R, (see, [1]). These results are further extensions of Mirbagheri-Ratliff-Rush's results in [8] and [13].

In [4], a regular ideal I, i.e., *grade* $I > 0$, for which $\tilde{I} = I$ is called a *Ratliff-Rush ideal*, and the ideal $\tilde{I}$ is called the *Ratliff-Rush ideal associated with* the regular ideal I. Subsequently, W. Heinzer et al. [6] introduced a concept analogous to this for modules over a commutative ring. Let us recall the following definition:

Definition 1. *(see, Heinzer et al. [6]). Let R be a commutative ring, let M be an R-module and let I be an ideal of R. The Ratliff-Rush closure of I w.r.t. M, denoted by $\widetilde{I_M}$, is defined to be the union of $(I^{n+1}M :_M I^n)$, where n varies in $\mathbb{N}$, i.e., $\widetilde{I_M} = \{e \in M : I^n e \subseteq I^{n+1}M$ for some $n \in \mathbb{N}\}$.*

If $M = R$, then the definition reduces to that of the usual Ratliff-Rush ideal associated to I in R (see, [12]). Furthermore, $\widetilde{I_M}$ is a submodule of M, and it is easy to see that $IM \subseteq \tilde{I}M \subseteq \widetilde{I_M}$. The ideal I is said to be *Ratliff-Rush closed* w.r.t. M if and only if $IM = \widetilde{I_M}$.

At the end of [6], the authors ask: *What conditions ensure that all suitably high powers of I are Ratliff-Rush closed w.r.t. M.* That is: When does the above condition (0.1) extend to Ratliff-Rush closure with respect to a module? This is answered in this paper.

Let R be a Noetherian ring and M a finitely generated R-module. For any ideal I of R, we denote by $G_R(I)$ (resp. $G_M(I)$) *the associated graded ring* $\oplus_{n\geq 0} I^n/I^{n+1}$ (resp. *the associated graded $G_R(I)$-module* $\oplus_{n\geq 0} I^nM/I^{n+1}M$). W. Heinzer et al. have shown in ([6, Fact 9] that there exists an element in the homogeneous ideal $\oplus_{n\geq 1} I^n/I^{n+1}$ that is a non-zerodivisor on the module $G_M(I)$ if and only if for all positive integers n, $\widetilde{I^n_M} = I^nM$. As a main result of this paper, we characterize, when all powers of an ideal I are Ratliff- Rush closed with respect to M in terms of the associated prime ideals of $G_M(I)$. More precisely we shall prove the following result:

Theorem 1. *(see, [9]) Let R be a commutative Noetherian ring, let M be a non-zero finitely generated R-module, and let I be an M-proper ideal of R such that grade $(I, M) > 0$. Then the following conditions are equivalent:*

(i) *All powers of I are Ratliff-Rush closed w.r.t. M.*

(ii) *For all $\mathfrak{p} \in Ass_{\mathscr{R}} \mathscr{M}/u\mathscr{M}, It \nsubseteq \mathfrak{p}$.*

(iii) *There exists an integer $k \geq 1$ and an element $x \in I^k$ such that $(I^{n+k}M :_M x) = I^nM$ for all integers $n \geq 1$.*

Throughout this paper, all rings considered will be commutative and will have non-zero identity elements. Such a ring will be denoted by R, and the terminology is, in general, the same as that in [2]. Let I be an ideal of R, and let M be a non-zero finitely generated module over R. We denote by $\mathcal{R}$ the *Rees ring* $R[u, It] := \oplus_{n\in\mathbb{Z}} I^n t^n$ of R w.r.t. I, where t is an indeterminate and $u = t^{-1}$. Also, the *graded Rees module* $M[u, It] := \oplus_{n\in\mathbb{Z}} I^n M$ over $\mathscr{R}$ is denoted by $\mathscr{M}$, which is a finitely generated graded $\mathscr{R}$-module. We shall say that I is *M-proper* if $M/IM \neq 0$, and, when this is the case, we define the *M-grade of I* (written *grade* (I, M)) to be the maximum length of all M-sequences contained in I. For any ideal I of R, *the radical of I*, denoted by $Rad(I)$, is defined to be the set $\{x \in R : x^n \in I$ for some $n \in \mathbb{N}\}$.

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