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Faithfulness of a class of modules with respect to the functor Hom (or Tensor) and related topics

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Abstract

Let R be a ring, $\alpha : \text{Mod-}R \to \text{Mod-}\mathbb{Z}$ be a functor and $\mathcal{C}$ be a non-empty class of R-modules. The class $\mathcal{C}$ is called *α-faithful* if $\alpha(X) \neq 0$ for all $0 \neq X$.
In this note, we state some recent results related to α-faithfulness for an arbitrary (certain) class of R-modules where $\alpha = \text{Hom}_R(M, -)$ or $M \otimes_R -$. We then consider some relevant open questions.

1 Modules with Hom-faithful lattice

- If M_R is module such that $\text{L}(M_R)$ = the set of all submodules of M_R, is $\text{Hom}_R(M, -)$-faithful then M was called *retractable* by **Khuri (1979)**. She use this concept to yielding a lattice isomorphism between certain submodules of M and right ideals of $\text{End}_R(M)$. See also [**Haghany-Vedadi (2005)**, endoprime modules.] and [**Haghany-Vedadi (2007)**, study of semi-projective retractable modules.]

Note: $\text{Hom}_{\mathbb{Z}}(\mathbb{Q}, X) = 0$ for all $X < \mathbb{Q}$.

The retractability is a necessary condition for defining prime and semiprime modules by Bican.

- **Bican (1980)**: Define the $*$-operation on $\text{L}(M_R)$ by $N*K = \text{Hom}_R(M, N)K$. Then M_R is called prime (semiprime) if $N * K = 0$ (resp. $N * N = 0$) implies $N = 0$ or $K = 0$ (resp. $N = 0$).
- **Lomp (2005)**: Open problem " *when semiprime modules are subdirect products of primes?*".
- **Dehghani-Vedadi (2015)**: Over many rings including commutative rings, semiprime modules are subdirect products of primes and the converse is true

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precisely if semiprime modules are enveloping for Mod-R.

Where do retractable modules appear?
- **Zelmanowitz (1981)**: Over a semiprime(reduced) ring, all torsionless modules are retractable.
- **Smith (2005)**: Over a commutative Noetherian, all projective module are retractable.
- **Haghany-Karamzadeh-Vedadi (2009)**: Over a commutative ring, all finitely generated modules are retractable.
- **Tolooei-Vedadi (2013)**: All are module are retractable iff all torsion theories on R are hereditary. If further, $R/J(R)$ is reduced and $J(R) \subset \mathrm{Cent}(R)$, then all R-modules are retractable iff R is a semi-Artinian ring (i.e., every factor ring of R has a minimal (right) ideal).
This is a generalization of a result due to **Ohtake (1981)**.
- A result similar to the above result are obtained by **Abyzov (2012)** for duo rings.

2 Hom-reversible and Hom(Tensor)-faithful modules

For any module M_R the smallest full subcategory of Mod-R generated by M which is closed under direct sums, factor modules and submodules is denoted by $\sigma[M_R]$ [14]. A module M_R is called:
Hom-faithful (resp. *tensor-faithful*), if $\sigma[M_R]$ is faithful w.r.t. the functor $\mathrm{Hom}_R(M, -)$ (resp. $M \otimes_R -$).
Hom-reversible: $\mathrm{Hom}_R(X, Y) = 0 \Rightarrow \mathrm{Hom}_R(Y, X) = 0$ for all $X, Y \in \sigma[M_R]$.

- Hom-reversible $\Rightarrow$ Hom-faithful.

-**Tolooei-Vedadi (2014)**: R_R is Hom-reversible iff R is a right perfect ring.
-**Haghany-Tolooei-Vedadi (2015)**: Let R be a ring Morita equivalent to commutative ring. Then (a) R is semi-Artinian if and only if all nonzero R-modules are Hom-faithful.
(b) The following are equivalent.
(1) R is a max ring.
(2) All nonzero R-modules are Tensor-faithful.
(3) $\forall\ 0 \neq N \leq M_R,\ \ \mathrm{ann}_R(N)M \neq M$.
(4) $\forall\ 0 \neq M_R$, $\mathrm{Hom}_R(M/J(M), M) \neq 0$.
(5) $\forall\ 0 \neq M_R,\ \ \exists\ N \leq M_R$ such that R/I has minimal ideal where $I =$

$\mathrm{ann}_R(M/N)$.
(6) $\forall\ 0 \neq M_R$, L(M_R) is faithful w.r.t $M \otimes_R -$.

3 Weak generator and T-nilpotence

Definition: An R-module M_R is said to be *weak-generator (w.g. for short)* for a class $\mathcal{C}$ if the class $\mathcal{C}$ is faithful w.r.t $\mathrm{Hom}_R(M, -)$. A w.g. for Mod-R is simply called *weak generator*.

Smith (2005). All R-modules are w.g. iff R is a semi-Artinian max-ring with unique simple R-module up to isomorphisms.

For any class $\mathcal{C}$, let $\mathcal{F}(C)$ = the collection of proper right ideals E of R such that the R-module $R/E \in \mathcal{C}$. **Smith-Vedadi (2006)**:
- M_R is a weak generator for a class $\mathcal{C} \Rightarrow$ the R-module R/A is a weak generator for $\mathcal{C}$, where $A = \mathrm{ann}_R(M)$.

- For any ideal I of R and any class $\mathcal{C}$ of R-modules.
The R-module R/I is a w.g. for $\mathcal{C} \Longleftrightarrow$ For each $E \in \mathcal{F}(C)$ and each sequence $a_1, a_2, ...$ in I there exists $n \geq 1$ such that $a_1 a_2 ... a_n \in E$.

Definition: A non-empty $Y \subseteq R$ is called *left T-nilpotent* provided for each sequence $y_1, y_2, ...$ in Y there exists $n \geq 1$ such that $y_1 y_2 ... y_n = 0$.
- Let R be a ring Morita equivalent to commutative ring.
(i) A finitely generated R-module M is w.g $\Longleftrightarrow$ $\mathrm{ann}_R(M)$ is T-nilpotent.
(ii) If R is semiprime Noetherian, then:
an arbitrary nonzero R-module M is w.g $\Longleftrightarrow$ $\sum_{f:M\to R} f(M)$ is an essential ideal in R.
(iii) *Nice Corollary*: An ideal I is T-nilpotent iff the ideal I[x] is T-nilpotent iff the ideal $\mathrm{Mat}_n(I)$ is T-nilpotent ($n \geq 1$).

4 some questions

$\mathbf{Q_1}$: What are module M_R such that Mod-R is faithful w.r.t the functor $M \otimes_R -$?
Note1: If R is commutative, we know that w.g modules have such property, due to the isomorphism $\mathrm{Hom}_R(M \otimes_R X, Y) \simeq \mathrm{Hom}_R(M, \mathrm{Hom}_R(X, Y))$.
$\mathbf{P_1}$: Let $\mathcal{C}$ be α-faithful. Find a minimum subclass $\mathcal{C}'$ of $\mathcal{C}$ such that the faithfulness of $\mathcal{C}$ is equal to the faithfulness of $\mathcal{C}'$.

Smith-Vedadi (2006): Let R be a (right) strongly prime ring (i.e. R embeds in $I^{(m)}$ where $I \leq R$), then M_R is a w.g for injective R-modules if and only if $\text{Hom}_R(M,\text{E}(R_R)) \neq 0$.

Note2: If $N \leq M$ and $\mathcal{C}$ is faithful w.r.t $\text{Hom}(M/N,-)$ (resp. $(M/N) \otimes_R -$) then $\mathcal{C}$ is faithful w.r.t $\text{Hom}(M,-)$ (resp. $M \otimes_R -$).

We say that a class $\mathcal{C}$ is *critical faithful* w.r.t $\text{Hom}(M,-)$ (resp. $M \otimes_R -$) if $\mathcal{C}$ is faithful w.r.t the functor but it is not faithful when M is replaced with a proper factor M/N.

$\mathbf{Q_2}$: For a given class $\mathcal{C}$, characterize modules M such that $\mathcal{C}$ is critical faithful w.r.t $\text{Hom}(M,-)$ (resp. $M \otimes_R -$).

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