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Annihilators and attached primes of local cohomology modules

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Let R be an arbitrary commutative Noetherian ring (with identity), $\mathfrak{a}$ an ideal of R and let M be a finitely generated R-module. An important problem concerning local cohomology is determining the annihilators of the ith local cohomology module $H^i_{\mathfrak{a}}(M)$. This problem has been studied by several authors; see for example [6], [8], [9], [10], [11]. Recently, in [2] Bahmanpour et al., proved an interesting result about the annihilator $\mathrm{Ann}_R(H^d_{\mathfrak{m}}(M))$, in the case $(R, \mathfrak{m})$ is a complete local ring of dimension d. We determine the annihilators of the top local cohomology module $H^{\dim M}_{\mathfrak{a}}(M)$. More precisely, we shall prove the following theorem which is a generalization of the main result of [2, Theorem 2.6] for an arbitrary ideal $\mathfrak{a}$ of an arbitrary Noetherian ring R.

Theorem 1.1. (cf. [1].) *Let R be a Noetherian ring and let M be a finitely generated R-module. Then for any ideal $\mathfrak{a}$ of R,* $\mathrm{Ann}_R(H^{\dim M}_{\mathfrak{a}}(M)) = \mathrm{Ann}_R(M/T_R(\mathfrak{a}, M))$*, where $T_R(\mathfrak{a}, M)$ denotes the largest submodule of M such that* $\mathrm{cd}(\mathfrak{a}, T_R(\mathfrak{a}, M)) < \mathrm{cd}(\mathfrak{a}, M)$.

Several corollaries of this result are given. A typical result in this direction is the following, which is a generalization of the main results of [2, Theorem 2.6] and [7, Theorem 2.4] for an ideal $\mathfrak{a}$ in an arbitrary Noetherian ring R.

Corollary 1.2. *Let R be a Noetherian ring and $\mathfrak{a}$ an ideal of R. Let M be a non-zero finitely generated R-module of finite dimension c such that* $\mathrm{cd}(\mathfrak{a}, M) = c$. *Then*

(i) $\mathrm{Ann}_R(H^c_{\mathfrak{a}}(M)) = \mathrm{Ann}_R(M/H^0_{\mathfrak{b}}(M)) = \mathrm{Ann}_R(M/\bigcap_{\mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}_j)=c} N_j)$*, where* $0 = \bigcap_{j=1}^n N_j$ *denotes a reduced primary decomposition of the zero submodule 0 in M and N_j is a $\mathfrak{p}_j$-primary submodule of M, for all $j = 1, \ldots, n$ and* $\mathfrak{b} := \Pi_{\mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}_j)\neq c}\mathfrak{p}_j$.

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(ii) *If* $\operatorname{Ass}_R M \subseteq \{\mathfrak{p} \in \operatorname{Supp} M \mid \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = c\}$, *then* $\operatorname{Ann}_R(H^c_{\mathfrak{a}}(M)) = \operatorname{Ann}_R(M)$.

(iii) $\operatorname{Rad}(\operatorname{Ann}_R(H^c_{\mathfrak{a}}(M))) = \bigcap_{\mathfrak{p}\in \operatorname{Ass}_R M,\, \operatorname{cd}(\mathfrak{a},R/\mathfrak{p})=c} \mathfrak{p} = \bigcap_{\mathfrak{p}\in \operatorname{Ass}_R(M/T_R(\mathfrak{a},M))} \mathfrak{p}$.

(iv) $V(\operatorname{Ann}_R(H^c_{\mathfrak{a}}(M))) = \operatorname{Supp}(M/T_R(\mathfrak{a}, M))$.

One of the basic problems concerning local cohomology is finding the set of attached primes of $H^i_{\mathfrak{a}}(M)$. We obtain some results about the attached primes of local cohomology modules. In fact, we derive the following consequence of Theorem 1.1 and Corollary 1.2, which provides an upper bound for the attached primes of $\operatorname{Att}_R H^{\operatorname{cd}(\mathfrak{a},M)}_{\mathfrak{a}}(M)$. This will generalize the main results of [4] and [3].

Theorem 1.3. *Let R be a Noetherian ring and $\mathfrak{a}$ an ideal of R. Let M be a finitely generated R-module such that $c := \operatorname{cd}(\mathfrak{a}, M)$ is finite. Then*

$$\operatorname{Att}_R H^c_{\mathfrak{a}}(M) \subseteq \{\mathfrak{p} \in \operatorname{Supp} M \mid \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = c\}.$$

Moreover, if $c = \dim M$, then

$$\operatorname{Att}_R H^c_{\mathfrak{a}}(M) = \{\mathfrak{p} \in \operatorname{mAss}_R M \mid \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = c\}.$$

For an R-module A, a prime ideal $\mathfrak{p}$ of R is said to be *attached prime to A* if $\mathfrak{p} = \operatorname{Ann}_R(A/B)$ for some submodule B of A. We denote the set of attached primes of A by $\operatorname{Att}_R A$. This definition agrees with the usual definition of attached prime if A has a secondary representation.
The second our main result is to give a complete characterization of the attached primes of the local cohomology module $H^{\dim R-1}_{\mathfrak{a}}(R)$. More precisely, we shall show the following result, which is an extension, as well as, a correction of the main theorem of [5].

Theorem 1.4. *Let $(R, \mathfrak{m})$ be a local (Noetherian) ring of dimension d. Let $\mathfrak{a}$ be an ideal of R such that $\dim R/\mathfrak{a} = 1$ and $H^d_{\mathfrak{a}}(R) = 0$. Then $\operatorname{Assh}_R R \subseteq \operatorname{Att}_R H^{d-1}_{\mathfrak{a}}(R)$. Moreover, if R is complete, then*

$$\operatorname{Att}_R H^{d-1}_{\mathfrak{a}}(R) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \dim R/\mathfrak{p} = d-1 \text{ and } \operatorname{Rad}(\mathfrak{a}+\mathfrak{p}) = \mathfrak{m}\} \cup \operatorname{Assh}_R R.$$

One of our tools for proving Theorem 1.4 is the following:

Theorem 1.5. *Let R be a Noetherian ring of finite dimension d and $\mathfrak{a}$ an ideal of R such that $H^d_{\mathfrak{a}}(R) = 0$. Then*

$$\mathrm{Att}_R H^{d-1}_{\mathfrak{a}}(R) = \{\mathfrak{p} \in \mathrm{Spec}\, R |\, \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = d-1\}.$$

For any R-module M, the ith local cohomology module of M with support in $V(\mathfrak{a})$ is defined by

$$H^i_{\mathfrak{a}}(M) := \varinjlim_{n\geq 1} \mathrm{Ext}^i_R(R/\mathfrak{a}^n, M).$$

The cohomological dimension of M with respect to $\mathfrak{a}$ is defined as

$$\mathrm{cd}(\mathfrak{a}, M) := \sup\{i \in \mathbb{Z} |\, H^i_{\mathfrak{a}}(M) \neq 0\}.$$

For each R-module L, we denote by $\mathrm{Assh}_R L$ (resp. $\mathrm{mAss}_R L$) the set $\{\mathfrak{p} \in \mathrm{Ass}_R\, L : \dim R/\mathfrak{p} = \dim L\}$ (resp. the set of minimal primes of $\mathrm{Ass}_R\, L$).

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